

12.1 Let us consider the following initial value problem on $(\mathbb{R}^{n+1}, \eta)$ (with $(t, x^1, \dots, x^n)$ denoting the usual Cartesian coordinate system):

$$\begin{cases} \square_{\eta} \psi = -(\partial_t \psi)^2, \\ \psi|_{t=0} = \psi_0, \quad \partial_t \psi|_{t=0} = \psi_1, \quad \text{for } \psi_0, \psi_1 \in C^\infty(\mathbb{R}^n). \end{cases} \quad (1)$$

- (a) Show that, if the above initial value problem has a smooth solution on a domain of the form $\{-T_1 \leq t \leq T_2\}$, then it is unique. (*Hint: Consider the equation satisfied by the difference $w = \psi^{(1)} - \psi^{(2)}$ of two solutions; recast it as an equation which is linear in terms of w and apply an energy estimate.*)
- (b) Show that the initial value problem (1) satisfies the domain of dependence property: If $(\psi_0, \psi_1) = (0, 0)$ on a ball $B_{x_0, R} = \{x \in \mathbb{R}^n : |x - x_0| \leq R\}$, then ψ vanishes on the domain of dependence of $B_{x_0, R}$, i.e. on the diamond domain $\Omega = \{(t, x) : |x - x_0| \leq R - |t|\}$ (you can assume that any solution, if it exists, is sufficiently smooth; this is something that we will prove in class).
- (c) Show that for any initial data of the form $(\psi_0, \psi_1) = (0, c)$, where $c \neq 0$ is a constant, the corresponding solution of (1) blows up in finite time (i.e. $\limsup_{t \rightarrow T^-} \|\psi(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} = +\infty$ for some $T \in \mathbb{R} \setminus \{0\}$) (*Hint: Observe that, for a solution which is constant in x , the above equation becomes a second order ODE in terms of t .*)
- (d) In the case $n \geq 3$, show that, for any $\epsilon > 0$, there exist *compactly supported* initial data (ψ_0, ψ_1) with $\|\psi_0\|_{H^1} + \|\psi_1\|_{L^2} < \epsilon$ which blow up in finite time. As a result, small initial energy is not enough to guarantee that (1) has a solution which exists globally in time; this is related to the fact that (1) is *supercritical* with respect to the energy norm. (*Hint: Consider the initial data from part (c) and cut them off in a ball of suitable radius, using the domain of dependence property to deduce the behaviour of the solution on the domain of dependence of that ball.*)

Remark. In the case where $n = 3$ (that is, the physical space dimension), it was shown by Fritz John that *every* smooth and compactly supported initial data set (ψ_0, ψ_1) for (1) that doesn't vanish identically gives rise to a solution that blows up in finite time.

12.2 In this exercise, we will establish Hardy's inequality. Like Poincaré's inequality, this inequality is used to control lower order norms of a function f in terms of norms of higher derivatives, albeit in unbounded domains.

- (a) Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be a C^1 function. Show that, for any $a > -1$, there exists a constant $C_a > 0$ depending only on a such that

$$\int_0^{+\infty} x^a |f|^2 dx \leq C_a \left(\int_0^{+\infty} x^{a+2} |f'|^2 dx + \limsup_{x \rightarrow +\infty} (x^{a+1} |f|^2(x)) \right).$$

In particular, if f is compactly supported, we have

$$\int_0^{+\infty} x^a |f|^2 dx \leq C_a \int_0^{+\infty} x^{a+2} |f'|^2 dx.$$

(Hint: Starting from the left hand side, write $x^a = \frac{d}{dx} \left(\frac{1}{a+1} x^{a+1} \right)$ and integrate by parts; you might want to use a Cauchy–Schwarz inequality at some point.)

- (b) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^1 function. Show that, for any $a > -n$, there exists a constant $C_{a,n} > 0$ depending only on a, n such that

$$\int_{\mathbb{R}^n} |x|^a |f|^2 dx \leq C_{a,n} \left(\int_{\mathbb{R}^n} |x|^{a+2} |\nabla f|^2 dx + \limsup_{R \rightarrow +\infty} \int_{|x|=R} R^{a+1} |f|^2 d\sigma \right).$$

(Hint: Use polar coordinates.)

- (c) Let ψ be a smooth solution of the wave equation

$$\square_\eta \psi = 0$$

on $(\mathbb{R}^n, \eta)$, $n \geq 3$, with smooth and compactly supported initial data (ψ_0, ψ_1) at $t = 0$. Show that

$$\sup_{T \in \mathbb{R}} \int_{t=T} \frac{1}{|x|^2} |\psi|^2 dx \leq C \mathcal{E}[\psi](0) < +\infty$$

for some constant $C > 0$ depending only on the dimension n , where $E[\psi](0)$ is the initial energy of ψ . Compare the above bound with Exercise 10.2.b.

12.3 In this exercise, we will show that solutions of the standard wave equation on Minkowski space-time decay polynomially fast towards timelike infinity. We will restrict ourselves to $(\mathbb{R}^{3+1}, \eta)$, even though similar arguments can be applied to the case of $\mathbb{R}^{n+1}$ for any $n \geq 3$. In what follows, we will denote with ϕ a given solution of

$$\square_\eta \phi = 0$$

on $\mathbb{R}^{3+1}$ emanating from smooth and compactly supported initial data at $t = 0$. We will denote by $r = |x|$ the usual spatial radius function, and by

$$u = t - r, \quad v = t + r$$

the usual double null coordinate functions.

- (a) Show that, in the polar coordinates (t, r, ω) on $\mathbb{R}^{3+1} \setminus \{r = 0\} \simeq \mathbb{R} \times (0, +\infty) \times \mathbb{S}^2$, the wave equation can be reexpressed in terms of $r\phi$ as

$$-\partial_t^2(r\phi) + \partial_r^2(r\phi) + \frac{1}{r^2} \Delta_{\mathbb{S}^2}(r\phi) = 0,$$

where $\delta_{\mathbb{S}^2}$ is the usual Laplace–Beltrami operator on the unit sphere $(\mathbb{S}^2, g_{\mathbb{S}^2})$; in the standard (θ, φ) coordinates on $\mathbb{S}^2$, it takes the form

$$\Delta_{\mathbb{S}^2} f = \frac{1}{\sin \theta} \partial_{\theta} (\sin \theta \partial_{\theta} f) + \frac{1}{\sin^2 \theta} \partial_{\varphi}^2 f.$$

In the (u, v, ω) coordinate system, we have

$$-4\partial_u \partial_v (r\phi) + \frac{1}{r^2} \Delta_{\mathbb{S}^2} (r\phi) = 0. \quad (2)$$

- (b) Let us now switch to the (u, v, ω) coordinate system. Show the following energy bounds along the leaves of the foliation $\{u = \text{const}\}$: For any $0 \leq \tau_1 \leq \tau_2$,

$$\int_{u=\tau_2} \left(4|\partial_v(r\phi)|^2 + \frac{1}{r^2} |\nabla_{\omega}(r\phi)|_{g_{\mathbb{S}^2}}^2 \right) dv d\omega \leq \int_{u=\tau_1} \left(4|\partial_v(r\phi)|^2 + \frac{1}{r^2} |\nabla_{\omega}(r\phi)|_{g_{\mathbb{S}^2}}^2 \right) dv d\omega \quad (3)$$

and

$$\int_{u=\tau_1} \left(4|\partial_v(r\phi)|^2 + \frac{1}{r^2} |\nabla_{\omega}(r\phi)|_{g_{\mathbb{S}^2}}^2 \right) dv d\omega \leq \int_{t=0} \left(|\partial_t(r\phi)|^2 + |\partial_r(r\phi)|^2 + \frac{1}{r^2} |\nabla_{\omega}(r\phi)|_{g_{\mathbb{S}^2}}^2 \right) dr d\omega,$$

where $|\nabla_{\omega}(r\phi)|_{g_{\mathbb{S}^2}}^2$ denotes the norm of the spherical gradient, evaluated with respect to the unit metric on the sphere, while $d\omega$ denotes the associated volume form on the unit sphere. The above is of course nothing more than the usual energy inequality, applied to domains bounded by the null cones.

(Hint: Multiply (2) with $\partial_t(r\phi) = (\partial_u + \partial_v)(r\phi)$ and integrate by parts over appropriate domains with respect to the volume form $du dv d\omega$. Note that you might have to truncate your domains by intersecting with $\{v \leq V\}$ and sending $V \rightarrow +\infty$.)

- (c) For any $p \in [0, 2]$, using $r^p \partial_v(r\phi)$ as a multiplier for (2), show that, for any $0 \leq \tau_1 \leq \tau_2$:

$$\begin{aligned} \int_{u=\tau_2} r^p |\partial_v(r\phi)|^2 dv d\omega + \int_{\tau_1 \leq u \leq \tau_2} r^{p-1} \left(|\partial_v(r\phi)|^2 + r^{-2} |\nabla_{\omega}(r\phi)|_{g_{\mathbb{S}^2}}^2 \right) du dv d\omega \\ \leq \int_{u=\tau_2} r^p |\partial_v(r\phi)|^2 dv d\omega. \end{aligned} \quad (4)$$

and

$$\int_{u=0} r^p |\partial_v(r\phi)|^2 dv d\omega \leq C \int_{t=0} r^p \left(|\partial_v(r\phi)|^2 + r^{-2} |\nabla_{\omega}(r\phi)|_{g_{\mathbb{S}^2}}^2 \right) dr d\omega$$

for some absolute constant $C > 0$ (i.e. independent of ϕ).

- (d) Show that if $f : [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function satisfying

$$\int_0^{+\infty} f(t) dt \leq C$$

then there exists a sequence of points $t_n \rightarrow +\infty$ with $t_{n+1} \in [2t_n, 4t_n]$ (we will call such a sequence *dyadic*) such that

$$f(t_n) \leq \frac{10C}{t_n}.$$

- *(e) Starting from (4) for $p = 2$ and using part (c), show that there exists a dyadic sequence τ_n and some absolute constant $C > 0$, such that

$$\int_{u=\tau_n} r |\partial_v(r\phi)|^2 dv d\omega \leq \frac{C}{\tau_n} \int_{t=0} r^2 \left(|\partial_v(r\phi)|^2 + r^{-2} |\nabla_\omega(r\phi)|_{g_{S^2}}^2 \right) dr d\omega.$$

Applying again (4) but this time for $p = 1$ on the intervals $\{\tau_n \leq u \leq \tau_{n+1}\}$, show that, for some $\tau'_n \in [\tau_n, \tau_{n+1}]$:

$$\int_{u=\tau'_n} \left(|\partial_v(r\phi)|^2 + r^{-2} |\nabla_\omega(r\phi)|_{g_{S^2}}^2 \right) dv d\omega \leq \frac{C'}{\tau_n^2} \int_{t=0} r^2 \left(|\partial_v(r\phi)|^2 + r^{-2} |\nabla_\omega(r\phi)|_{g_{S^2}}^2 \right) dr d\omega$$

for some absolute constant $C' > 0$.

- (f) Using the energy bound (3), deduce that, for any $\tau \geq 0$:

$$\int_{u=\tau} \left(|\partial_v(r\phi)|^2 + r^{-2} |\nabla_\omega(r\phi)|_{g_{S^2}}^2 \right) dv d\omega \leq \frac{C''}{\tau^2} \int_{t=0} r^2 \left(|\partial_v(r\phi)|^2 + r^{-2} |\nabla_\omega(r\phi)|_{g_{S^2}}^2 \right) dr d\omega$$

for some absolute constant $C'' > 0$. In particular, the energy flux through the cones $\{u = \tau\}$ decays like τ^{-2} as $\tau \rightarrow +\infty$.

- *(g) By commuting the wave equation with Cartesian derivatives, deduce an analogous decay estimate for higher order energy fluxes of ϕ . Applying the Sobolev inequality, deduce that

$$\phi(u, v, \omega) \lesssim u^{-1} \quad \text{and} \quad r\phi(u, v, \omega) \lesssim u^{-\frac{1}{2}},$$

with the constant implicit in the $\lesssim$ notation depending only on the initial data at $t = 0$.